Decay of a charged scalar field around a black hole: Quasinormal modes of RN, RNAdS, and dilaton black holes

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It is well known that the charged scalar perturbations of the Reissner-Nordström metric will decay slower at very late times than the neutral ones, thereby dominating in the late time signal. We show that, at the stage of quasinormal (QN) ringing, on the contrary, the neutral perturbations will decay slower for RN, RNAdS and dilaton black holes. The QN frequencies of the nearly extreme RN black hole have the same imaginary parts (damping times) for charged and neutral perturbations. An explanation of this fact is not clear but, possibly, is connected with the Choptuik scaling.

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I. INTRODUCTION

The response of a Schwarzschild black hole as a Gaussian wave packet impinges upon it consists of decaying quasinormal oscillations, dominating after time $t \approx 70M$, and inverse power-low tails, dominating after time $t \approx 300M$, where M is the black hole mass (see [1] and references therein). The quasinormal (QN) ringing can be caused by either external fields or by the formation of a black hole itself, and the characteristic frequencies do not depend on a form of perturbations, giving us a ''footprint'' of a black hole.

Becuase of AdS conformal field theory (CFT) correspondence [2] the investigation of the quasinormal frequencies of AdS black holes is appealing now: it gives the thermalization time scale for a field perturbation [3], namely, the imaginary part of the quasinormal frequency, being inversely proportional to the damping time of a given mode, determines the relaxation time of a field. Thus the more imaginary part of ω the faster a given field comes to an equilibrium.

The investigation of the QN modes within the AdS-CFT correspondence was initiated on the AdS gravity side by Horowitz and Hubeny in [3] for a massless scalar field. Then quasinormal modes associated with perturbations of different fields were considered in many works [4–10]. An exact expression for the three-dimensional Bañados-Teitelboim-Zanelli (BTZ) black hole QN modes corresponding to fields of different spin was obtained by Cardoso and Lemos in [4]. Recently similar work for the BTZ black hole was done on the CFT side [26].

On the AdS gravity side, it was found that for the neutral massless scalar field in the background of Reissner-Nordström-AdS (RNAdS) black hole with small charge, the more the black hole charge is, the quicker its approach to thermal equilibrium in CFT [8], and after the black hole charge approaches some critical value the situation changes to the contrary [9]. This repeats the behavior of the usual RN quasinormal spectrum, where the imaginary part of ω grows with the black hole charge up to some maximum, and then begins to decrease (at least for $l \geqslant 0$ [15]).

Summarizing the results of the papers [9], [11–13], one can see that the late time radiative behavior of a neutral scalar field for asymptotically flat (RN) and asymptotically (anti−)de Sitter (RNdS and RNAdS) black hole space-times is essentially different: in the first case the inverse power-law tails are dominating, while in the second it is an exponential decay. This decay is oscillatory for RNAdS, and for RNdS when a scalar field is strongly coupled to curvature.

When collapsing a charged matter, a charged black hole forms. Thus the evolution of a charged scalar field outside the RN black hole is most relevant. The late time behavior of a charged scalar field was considered by Hod and Piran [14]. There it was shown that in the radiative tails the neutral perturbations decay faster than the charged ones and therefore dominate at very late times. In addition, while at timelike and null infinity inverse power-law tails appear, along the future black hole event horizon, an oscillatory behavior accompanies this tail.

At present it is believed on different grounds, the main of which is the low energy limit of superstring theory, that a black hole possesses a specific scalar field called the dilaton. It drastically changes the properties of a black hole depending on the value of the dilaton coupling constant. At late times charged perturbations dominate as well outside such black holes [20]. Thus it would be interesting to cover this class of BH's in our investigation.

Thus there is a quite clear picture of asymptotic behavior of the radiation corresponding to charged perturbations, while its behavior during the stage of quasinormal ringing is lacking. This motivated us to study the behavior of a complex (charged) scalar field during the quasinormal ringing through the computing of its resonant characteristic frequencies for RN and RNAdS and dilaton black holes. In Sec. II we shall compute the quasinormal frequencies of the RN black hole for different multipole numbers l, in Sec. III the case of the RNAdS black hole is considered, and in Sec. IV the dilaton black QN frequencies are obtained. We have found that the modes of the nearly extremal RN black holes have the same damping times for charged and neutral perturbations. The possible connection of this fact with the critical collapse is discussed in Sec. V.

II. REISSNER-NORDSTRÖM BLACK HOLE

We shall consider the evolution of the charged scalar perturbations field in the background of the Reissner-Nordström metric:

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$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_2^2, \qquad (1)$$

where $f(r) = 1 - 2M/r + Q^2/r^2$. The wave equation of the complex scalar field has the form

$$\phi_{;ab}g^{ab} - ieA_a g^{ab}(2\phi_{;b} - ieA_b\phi) - ieA_{a;b}g^{ab}\phi = 0. \qquad (2)$$

Here the electromagnetic potential $A_t = C - Q/r$, and C is a constant. After representation of the charged scalar field into spherical harmonics and some algebra the equation of motion takes the form [14]

$$\psi_{,tt} + 2ie\frac{Q}{r}\psi_{,t} - \psi_{,r*r*} + V\psi = 0, \qquad (3)$$

where

$$V = f(r)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3} - \frac{2Q^2}{r^4}\right) - e^2\frac{Q^2}{r^2}, \qquad (4)$$

and $\psi = \psi(r)e^{-i\omega t}$, $dr^* = dr/f(r)$. One can compute the quasinormal frequencies stipulated by the above potential by using the third order WKB formula of Iyer and Will [16],

$$\frac{iQ_0}{\sqrt{2Q_0''}} - \Lambda(n) - \Omega(n) = n + \frac{1}{2}, \qquad (5)$$

where $\Lambda(n)$, $\Omega(n)$ are second and third order WKB correction terms depending on the potential V and its derivatives in the maximum. Here $Q = -V + \omega^2 - 2(eQ/r)\omega$. Since Q depends on ω, the procedure of finding of the QN frequencies is the following: one fixes all the parameter of the QN frequency, namely, the multipole index l, the overtone number n, the black hole mass and charge M and Q, and e; then one finds the value of r at which Q attains a maximum as a numerical function of ω and substituting it into the formula (5) one finds, with the trial and error way, ω, which satisfies Eq. (5). Note that Q is, generally, complex and we just continue Eq. (5) into the complex plane as it is prescribed in [16].

It is essential that for $l=0$ modes the WKB formula gives the worse precision: a relative error, for example, for scalar perturbations of Schwarzschild BH, may be of order 10 percent [17]. Nevertheless the more l (and the less n), the more accurate WKB formula is, and already for $l=3$, $n=0$, according to general experience, a relative error may be of order 10^{-2} percent. Thus in order to be sure that not only the WKB frequencies of charged and neutral perturbations coincide in the extremal limit, but also the true frequencies do the same, one needs to proceed computations to higher l. We can see the characteristic behavior of the QN spectrum from Table I and Figs. 1 and 2, where $l = 1,2$ and 3 fundamental $(n=0)$ QN frequencies corresponding to neutral and charged perturbations are presented. For higher multipole indexes, WKB precision is better.

The real part of ω for both neutral and charged scalar fields grows with increasing of charge Q; ω_{Im} is more for charged perturbations than for a neutral one. In addition, and this is the most interesting feature of charged QN spectrum, the imaginary part of a given "charged mode" approaches the neutral one in the limit of the extremal black hole. Within

TABLE I. The quasinormal frequencies for RN BH, $l=1,2$ $n=0$, $e=0$ (first line) and $e=0.1$ (second line).

Q	$l=1$	$l=2$
0	$0.2911-0.0980i$	$0.4832-0.0968i$
	$0.2911-0.0980i$	$0.4832-0.0968i$
0.1	$0.2916-0.0981i$	$0.4840-0.0969i$
	$0.2951-0.0984i$	$0.4874-0.0971i$
0.3	$0.2958-0.0984i$	$0.4908-0.0973i$
	$0.3064-0.0995i$	$0.5011-0.0979i$
0.5	$0.3049-0.0991i$	$0.5056-0.0980i$
	$0.3233-0.1008i$	$0.5236-0.0990i$
0.7	$0.3212-0.0996i$	$0.5322-0.0986i$
	$0.3490-0.1018i$	$0.5593-0.1000i$
0.8	$0.3337-0.0992i$	$0.5527-0.0983i$
	$0.3672-0.1015i$	$0.5855-0.0999i$
0.9	$0.3509-0.0972i$	$0.5815-0.0966i$
	$0.3918-0.0993i$	$0.6214-0.0980i$
0.95	$0.3622-0.0946i$	$0.6011-0.0945i$
	$0.4079-0.0960i$	$0.6459-0.0952i$
0.99	$0.3729-0.0907i$	$0.6205-0.0902i$
	$0.4231-0.0908i$	$0.6701-0.0904i$

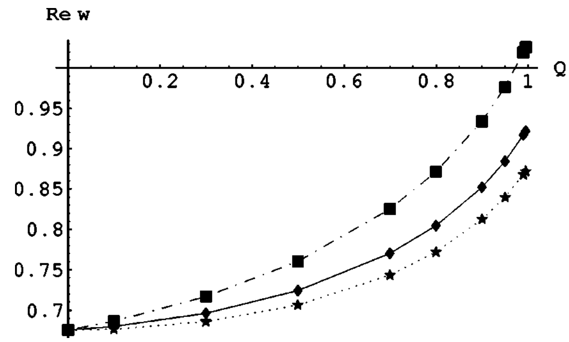

FIG. 1. Real part of ω, $l=3$, $n=0$, $e=0$ (star), $e=0.1$ (diamond) and $e=0.3$ (box), for Q, running from 0 to 0.995 (RN BH).

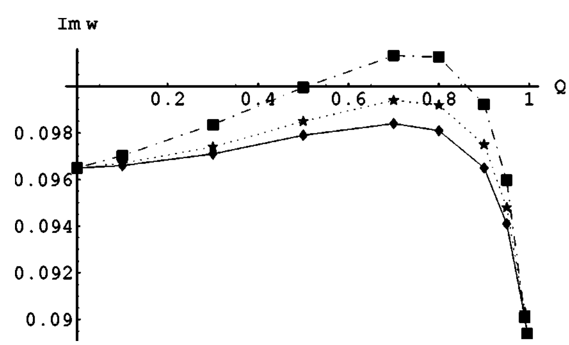

FIG. 2. Imaginary part of ω, $l=3$, $n=0$, $e=0$ (diamond), $e=0.1$ (star) and $e=0.3$ (box), for Q, running from 0 to 0.995 (RN BH). At $Q=0.995$ $\omega_{Im}=0.08938$ for $e=0$ and $\omega_{Im}=0.08943$ for $e=0.3$, the coincidence is within precision of the WKB method.

the third order WKB method one can check it with a high accuracy for higher multipole number perturbations.

III. REISSNER–NORDSTRÖM–ANTI–de SITTER BLACK HOLE

The Reissner–anti–de Sitter metric has the form

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2, \tag{6}$$

where

$$f(r) = 1 - \frac{r_+}{r} - \frac{r_+^3}{rR^2} - \frac{Q^2}{rr_+} + \frac{Q^2}{r^2} + \frac{r^2}{R^2}. \tag{7}$$

Here r_+ is an outer horizon, R is the AdS radius, and

$$M = \frac{1}{2}\left(r_+ + \frac{r_+^3}{R^2} + \frac{Q^2}{r_+} \right) \tag{8}$$

is the mass of the black hole.

Quasinormal oscillations associated with the decay of the charged scalar field in the background of RNAdS are governed by the wave equation (3), which can be transformed to the form

$$f(r)\frac{d^2\psi(r)}{dr^2} + (f'(r) - 2i\omega)\frac{d\psi(r)}{dr} - U(r)\psi(r) = 0. \tag{9}$$

Here the $U(r)$ is, again, a frequency dependent potential, determined by the formula

$$U(r) = \frac{f'(r)}{r} + \frac{l(l+1)}{r^2} + \frac{1}{f(r)}\left(2\frac{eQ}{r}\omega - e^2\frac{Q^2}{r^2} \right). \tag{10}$$

By rescaling of r we can put $R=1$. The effective potential $V(r) = f(r)U(r) - 2(eQ/r)\omega$ with respect to the wave equation (3) written in the tortoise coordinate r^* is infinite at spatial infinity. Thus the wave function is considered to vanish at infinity and satisfies the purely in-going wave condition at the black hole horizon. Then one can compute the quasinormal frequencies stipulated by the potential (10) following the procedure of Horowitz and Hubeny [3]. The main point of that approach is to expand the solution to the wave equation (9) around $x_+ = 1/r_+$ $(x = 1/r)$

$$\psi(x) = \sum_{n=0}^{\infty} a_n(\omega)(x - x_+)^n \tag{11}$$

and to find the roots of the equation $\psi(x=0)=0$ following from the boundary condition at infinity. In fact, one has to truncate the sum (11) at some large $n=N$ and check that for greater n the roots converge.

While the quasinormal modes of an asymptotically flat black hole are proportional to its mass, those of an asymptotically anti–de Sitter black hole depend upon the radius of a black hole. For large (r_+ is much greater than the anti–de Sitter radius R) and intermediate Schwarzschild–anti–de Sitter black holes, both ω_{Re} and ω_{Im} are roughly proportional to the black hole temperature. For small black holes $r_+ \ll R$ this linearity breaks and in the limit $r_+ \to 0$ the QNM approaches the pure anti–de Sitter modes [3,18]. Since it is a large black hole which is of direct interest for AdS-CFT correspondence, we shall restrict ourselves to this black hole regime.

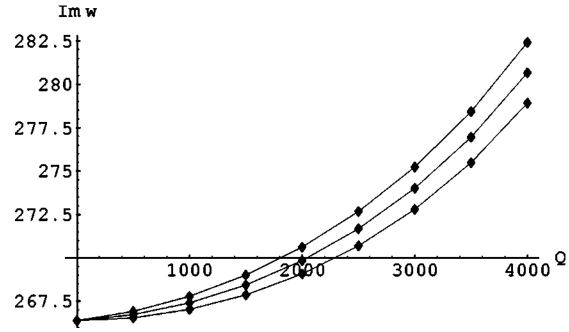

FIG. 3. Imaginary part of ω, $l=0$, $n=0$ for $e=0$, $e=5\times10^{-5}$, and $e=10^{-4}$ from the bottom to the top (RNAdS BH).

From Figs. 3 and 4 one can see that ω_{Re} and ω_{Im} grow with increasing of the charge conjugation e, i.e., the real oscillation frequency is more for charged perturbations than for neutral ones and the damping time of a given mode is more for neutral perturbations. Yet we managed to compute only the lowly charged case, due to the two difficulties. First, when Q and e grow, the number of terms in the truncated sum representing the wave function ψ increases: one has to sum over $N\sim10^3$ and more, and thus one has to guess new modes through the trial and error way. At the same time, the minimums of the truncated sum corresponding to the quasinormal frequencies are most narrow in the ω plane and one has to guess a lot of figures in the quasinormal frequency in order to catch the above minimum. Thus for the highly charged case one has to resort to another method of calculations of QN modes.

IV. DILATON BLACK HOLE

A wide class of theories includes the stationary spherically symmetric black hole solution with massless scalar field of some specific form (dilaton):

$$ds^2 = \lambda^2 dt^2 - \lambda^{-2}dr^2 - R^2 d\theta^2 - R^2\sin^2\theta d\varphi^2, \tag{12}$$

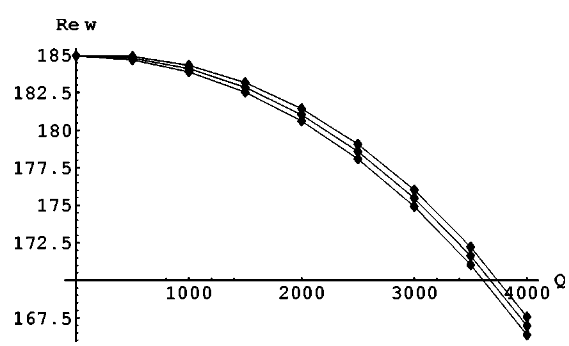

FIG. 4. Real part of ω, $l=0$, $n=0$ for $e=0$, $e=5\times10^{-5}$, and $e=10^{-4}$ from the bottom to the top (RNAdS BH).

where

$$\lambda^2=\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{(1-a^2/(1+a^2)},$$

$$R^2=r^2\left(1-\frac{r_-}{r}\right)^{2a^2/(1+a^2)}, \tag{13}$$

and

$$2M=r_+ + \left(\frac{1-a^2}{1+a^2}\right)r_-, \qquad Q^2=\frac{r_-r_+}{1+a^2}. \tag{14}$$

Here the dilaton and electromagnetic fields are given by the formulas

$$e^{2a\Phi}=\left(1-\frac{r_-}{r}\right)^{2a^2/(1+a^2)}, \qquad F_{tr}=\frac{e^{2a\Phi}Q}{R^2}, \tag{15}$$

where a is a non-negative dimensionless value representing coupling. The case $a=0$ corresponds to the classic Reissner-Nordström metric, the case $a=1$ is suggested by the low energy limit of the superstring theory, and $a=\sqrt{3}$ corresponds to the dimensionally reduced Kaluza-Klein black hole.

Following the above WKB method we shall compute here the quasinormal modes corresponding to the neutral and charged massless scalar test field. We do not consider the coupling of the scalar field outside the black hole with the dilaton, i.e. the scalar field simply propagates in the black hole background.

The wave function obeys Eq. (3) with the effective potential

$$V(r)=\frac{R,_{r*r*}}{R}+\frac{l(l+1)\lambda^2}{R^2}-e^2\frac{Q^2}{r^2}. \tag{16}$$

This potential is broadening near the extremal limit [19].

In the case of dilaton BH both real and imaginary parts of ω grow with the increase of either Q or e. Nevertheless the imaginary part of ω of the charged field does not approach that of the neutral one in the nearly extremal regime. One can see it on example of $l=3$, $n=0$ modes where the WKB method should give reasonable accuracy (see Table II). Yet there is a possibility that during the broadening of the effective potential of the nearly extremal dilaton BH [19], the inaccuracy in the WKB formula may increase. Thus it would be most relevant to compute the dilaton BH QN frequencies numerically.

Recently it has been obtained that at late times the neutral scalar field falls off faster than the charged field in the dilaton BH background [20,22]. Thus we see that for a dilaton black hole the situation changes on the contrary as well: domination of a neutral field during the quasinormal ringing and of a charged field at late times.

TABLE II. The quasinormal frequencies for $a=1$ dilaton black hole $l=3$, $n=0$, $e=0$ and $e=0.1$.

Q	$e=0$	$e=0.1$
0	$0.67521-0.09651i$	$0.67521-0.09651i$
0.2	$0.67977-0.09673i$	$0.68650-0.09970i$
0.4	$0.69411-0.09738i$	$0.70768-0.09799i$
0.6	$0.72045-0.09853i$	$0.74109-0.09941i$
0.8	$0.76379-0.10028i$	$0.79189-0.10138i$
1.0	$0.83580-0.10273i$	$0.87209-0.10398i$
1.1	$0.89116-0.10419i$	$0.93205-0.10549i$
1.2	$0.97122-0.10559i$	$1.01734-0.10691i$
1.3	$1.10340-0.10582i$	$1.15602-0.10711i$
1.35	$1.21774-0.10357i$	$1.27473-0.10478i$
1.4	$1.45670-0.08874i$	$1.52076-0.08969i$

V. DISCUSSION

We have learned here that the damping time of the quasinormal oscillations associated with a charged scalar field in the background of the Reissner-Nordström black hole is less than that of a neutral one. Thus, that is the neutral perturbation which will dominate at later stages of quasinormal ringing. Yet, we know that at late times the charged perturbations are dominating, and one could expect, possibly, that the same sort of perturbations must dominate in the earlier stages of radiation. The logic of the process, however, is different. As was shown in [14], the late time behavior of the charged scalar field is entirely determined by the flat space-time effects, while that of the neutral perturbations is dependent on the relation between the "tortoise" $r*$ coordinate and r, i.e., by the space-time curvature. In other words, the radiative tail of the charged field arises due to the backscattering of this field off the electromagnetic potential far away from the black hole, while in the case of the neutral fields it is the effects of gravitation near the black hole (curvature effects). In this context it seems natural that in the earlier periods of radiation (quasinormal ringing) the curvature effects are dominating, and the neutral perturbations will damp slower.

Another interesting point of this study is the coincidence of the imaginary parts of ω for charged and neutral perturbations for the nearly extremal black hole. It is seen at once that since the universal index appearing in the phenomena of critical collapse β equals 0.37 both for charged [23] and neutral [24] scalar fields, then there may be a connection between the behavior of the quasinormal spectrum of nearly extremal black holes and the critical exponent for a given black hole. Yet the latter conjecture seems be too strong, and the black hole quasinormal modes may be related to the critical exponent in some specific space-time geometries [3,25].

In this connection it is interesting to recall that the nearly extremal RN black hole is effectively described by the AdS_2 black hole after spherically symmetric dimensional reduction [27]. For such a reduced nearly extremal black hole an exact relation between the quasinormal modes and the critical exponent is obtained in [25]. The possible connection of the QNM and critical collapse for the three dimensional BTZ BH is discussed in [26].

Yet the coincidence of the damping times for charged and neutral modes of the nearly extremal black hole is, apparently, an exclusive property of RN BH, and is not appropriate to other black holes. It is possible also that this coincidence takes place only for a massless scalar field, since for a massive one the situation is qualitatively different [28].

Thus any kind of satisfactory explanation of the above coincidence from a physical point of view is lacking.

Whether the damping times of charged and neutral perturbations in the nearly extremal limit will coincide for RNAdS BH, and for an asymptotically nonflat black hole in general, is a question for further investigation.

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